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A METHOD OF DETERMINING THE PIEZOELECTRIC MODULUS OF A NONUNIFORMLY POLARIZED ROD

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A method is proposed for determination of the depolarization function of a rod of piezoelectric ceramics for a specified amplitude–frequency characteristic of the current. The problem is reduced to a nonlinear integral equation solved by means of a combination of Tikhonov's linearization and regularization methods. The uniqueness of the solution is shown and a series of numerical experiments is carried out with the aim to determine the polarization law.

In the recent past, devices made of piezoelectric materials with inhomogeneous polarization have widely been introduced in industry [1, 2]. In this connection, one of the main problems is to determine the characteristics of the material that depend on the coordinates, among which piezoelectric moduli possess the maximum variability.

In the present paper, a method of determining the relation between the variation in the piezoelectric modulus $d_{31}(x_1)$ and the vibration frequency, which is varied in a certain range, is proposed for the case where the amplitude value of the current is specified. The problem is reduced to a nonlinear integral equation of the first kind, the solution of which is constructed by means of a combination of the linearization and regularization methods proposed by Tikhonov.

1. We consider vibrations of a piezoelectric ceramic rod oriented along the Ox_1 axis with electrodes located at the surfaces perpendicular to the x_3 axis. We assume that the length of the rod l is much greater than its thickness h and width b. The problem is then one-dimensional, and the stress-tensor components σ_{ij} and the vector of electric intensity E_i can be assumed to be independent of the coordinates x_2 and x_3. Since $E_1 = E_2 = 0$ at the surfaces with electrodes, one can set $E_1 = E_2 = 0$ at each point of the rod, provided that $h/l \ll 1$. From all the stress-tensor components, the only nonzero component is σ_{11}. Moreover, we assume that owing to the partial depolarization of the rod [for example, as a result of the action of a nonuniform heating field with a temperature higher than the Curie point, the end $x_1 = l$ can be depolarized completely, i.e., $d_{31}(l)$ vanishes], the piezoelectric modulus d_{31} is no longer constant, and it is a certain decreasing function of the coordinate $d_{31} = d_{31}(x_1)$. In addition, we assume that the modulus of elasticity and the dielectric permittivity are constant. In this case, the equations of state [3] have the form

$$\frac{\partial u_1}{\partial x_1} = s_{11}^E \sigma_{11} + d_{31}(x_1) E_3, \qquad D_3 = d_{31}(x_1)\sigma_{11} + \epsilon_{33}^{\sigma} E_3 \tag{1.1}$$

(ϵ is the dielectric constant) and, from the electroelasticity equations, we have only one equation

$$\frac{\partial \sigma_{11}}{\partial x_1} = \rho \frac{\partial^2 u_1}{\partial t^2}, \tag{1.2}$$

since the equation $\mathrm{div} D = 0$ is satisfied identically.

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Assuming that the ends of the rod $x_1 = 0$ and l are stress-free, we find a relation between the current $I(t) = -\frac{\partial}{\partial t}\int\limits_s D_3\,dx$ and the electric-field strength $E_3(x,t) = E(t)$.

Eliminating the displacement $u_1(x_1,t)$ from (1.1) and (1.2), we obtain the boundary-value problem

$$\frac{\partial^2\sigma_{11}}{\partial x_1^2} = \frac{1}{c^2}\frac{\partial^2\sigma_{11}}{\partial t^2} + \rho d_{31}(x_1)\frac{\partial^2 E}{\partial t^2}, \qquad \sigma_{11}\Big|_{x=0,l} = 0, \qquad \sigma_{11}\Big|_{t<0} = 0, \tag{1.3}$$

where $c^2 = 1/(\rho s_{11}^E)$.

We formulate an inverse problem: to determine the piezoelectric modulus $d_{31}(x_1)$ for a specified current $I(t)$, where $0 \leqslant t \leqslant T$, or its amplitude component as a function of vibration frequency. A widespread method of solving these problems in the nonstationary case is the method of reducing them to Volterra integral equation [4, 5].

Remark. Formulation of the inverse problem of determining the function for given displacements at one end of the rod is well studied [4, 5]. In practice, however, the measurement of the amplitude–frequency characteristic of the current is a much simpler experiment compared to the measurement of the amplitude–frequency characteristic of the displacement.

We derive a governing relation for determination of $d_{31}(x_1)$ for steady vibrations of the rod.

2. We consider the simplest case where the excitation of the rod is harmonic, i.e., $E(t) = E_0 e^{i\omega t}$ and $I(t) = I_0 e^{i\omega t}$. We find the amplitude of the current I_0 and the conductance of the rod Z:

$$\frac{1}{Z} = \frac{I_0}{E_0 h} = -iB_0\Omega R(\Omega); \tag{2.1}$$

$$R(\Omega) = 1 - \frac{k^2}{\sin\Omega}F(\Omega),$$

$$F(\Omega) = \Omega\int\limits_0^1 f(y)\Big[\int\limits_0^1 f(\eta)\sin(\Omega(\eta-1))\,d\eta\,\sin(\Omega y) - \int\limits_0^y \sin(\Omega(\eta-y))f(\eta)\,d\eta\,\sin\Omega\Big]\,dy, \tag{2.2}$$

$$\Omega \in [\Omega_1,\Omega_2].$$

Here the following dimensionless parameters and functions are introduced:

$$\Omega = \frac{\omega l}{c}, \quad B_0 = \frac{\epsilon_{33}^\sigma bc}{h}, \quad k^2 = \frac{d_{31}^2(0)}{\epsilon_{33}^\sigma s_{11}^E}, \quad f(y) = \frac{d_{31}(x_1)}{d_{31}(0)}, \quad y = \frac{x_1}{l}.$$

It is well known that the condition $R(\Omega) \to \infty$ determines the resonance frequencies of the system:

$$\sin\Omega = 0 \quad\Rightarrow\quad \Omega_n = \pi n, \qquad \omega_n = \frac{\pi n}{l\sqrt{\rho s_{11}^E}}, \qquad n = 1,2,\ldots .$$

We note that these frequencies do not depend on the variation in $d_{31}(x_1)$, and depolarization does not alter the resonances of the system.

The condition $R(\Omega) = 0$ determines the antiresonance frequencies, which depend strongly on $d_{31}(x_1)$. The equation for antiresonance frequencies has the form

$$\sin\Omega - k^2\Omega\int\limits_0^1 f(y)\Big[\int\limits_0^1 \sin(\Omega(\eta-1))f(\eta)\,d\eta\,\sin(\Omega y) - \int\limits_0^y \sin(\Omega(\eta-y))f(\eta)\,d\eta\,\sin\Omega\Big]\,dy = 0$$

and can be used for obtaining information on the function $d_{31}(x_1)$.

We formulate the inverse problem of determining the function $f(y) \in U = L_2[0,1] \cap M[0,1]$ [6]. Here $L_2[0,1]$ is the space of square-summable functions on $[0,1]$, and $M[0,1]$ is the space of positive functions decreasing monotonically on $[0,1]$, which is attributed to the physical properties of the desired function $f(y)$.

As the initial data, we use the function $F(\Omega) = F_*(\Omega)$, which is assumed to be defined on the interval $\Omega \in [\Omega_1,\Omega_2]$ and can be expressed in terms of the amplitude characteristic of the current $\Omega R(\Omega)$ from relation (2.2).

Thus, this inverse problem reduces to the nonlinear integral equation for the desired function $f(y) \in U$:

$$A(f) = \Omega \int_0^1 f(y)\Big[\int_0^1 f(\eta)\sin(\Omega(\eta-1))\,d\eta\,\sin(\Omega y)$$

$$-\int_0^y \sin(\Omega(\eta-y))f(\eta)\,d\eta\,\sin\Omega\Big]\,dy = F_*(\Omega), \quad \Omega \in [\Omega_1,\Omega_2]. \tag{2.3}$$

We investigate the question of uniqueness of the solution of the integral equation (2.3). We show that, for certain constraints imposed on the interval $[\Omega_1,\Omega_2]$, its solution is unique in U. We suppose the contrary, i.e., Eq. (2.3) admits two distinct solutions $f_1, f_2 \in U$. Next, we represent the operator $A(f)$ in $L_2[0,1]$ in the form of a scalar product $A(f) = (A_0 f, f)$, where A_0 is the linear operator with a symmetric kernel:

$$A_0(\Omega,y,\eta) = \Omega \begin{cases} \sin(\Omega(\eta-1))\sin(\Omega y), & y<\eta, \\ \sin(\Omega y)\sin(\Omega(y-1)), & y>\eta. \end{cases}$$

One can easily see that $(A_0 f, f) \leqslant 0$ at least for $0<\Omega<\pi$, the equality being possible only for $f=0$. Using this property and the condition $(A_0 f_1, f_1) = (A_0 f_2, f_2)$, we establish the inequalities

$$(A_0(f_1-f_2), f_1-f_2) = 2(A_0 f_1, f_1-f_2) \leqslant 0,$$

$$(A_0(f_1-f_2), f_1-f_2) = 2(A_0 f_2, f_2-f_1) \leqslant 0.$$

Since $A_0 f_1 > 0$ and $A_0 f_2 > 0$ for $0<\Omega<\pi$, the last two inequalities imply that $f_1 - f_2 = 0$. Thus, the uniqueness is established.

3. It is well known that the procedure of solution of the nonlinear equation (2.3) is an incorrect problem [7]; therefore, regularization algorithms should be used. We construct the solution of (2.3) by two stages on the basis of an approach that combines A. N. Tikhonov's linearization and regularization methods.

At the first stage, we construct the solution of Eq. (2.3) in the class of linear nonincreasing functions

$$f = f_0(y) = a_0 + a_1 y; \tag{3.1}$$

moreover, from the physical considerations, we obtain the following constraints imposed on the constants a_0 and a_1: $0 \leqslant a_0 \leqslant 1$, $a_1 \leqslant 0$, and $a_0 + a_1 \geqslant 0$, which determine a triangle U_0 on the (a_0, a_1) plane.

The constants a_0 and a_1 are found from the minimum condition for the nonquadratic functional

$$\Phi = \int_{\Omega_1}^{\Omega_2} |A(f_0) - F|^2\,d\Omega$$

on a set of *a priori* constraints U_0.

We note that, in this case, the integrals in the expression for the operator $A(f_0)$ are readily calculated and $A(f_0) = F_0(\Omega)$, where $F_0(\Omega) = a_0^2\alpha_0(\Omega) + a_0 a_1 \alpha_1(\Omega) + a_1^2\alpha_2(\Omega)$ and the following notation is introduced:

$$\alpha_0(\Omega) = -2\sin(\Omega/2)\Big(\frac{2\sin(\Omega/2)}{\Omega} - \cos(\Omega/2)\Big) = \alpha_1(\Omega), \qquad \alpha_2(\Omega) = -\frac{\sin\Omega}{\Omega^2} + \frac{1}{3}\sin\Omega + \frac{\cos\Omega}{\Omega}.$$

Thus, we have a problem of determining the minimum of the function $\Phi(a_0, a_1)$ in the domain U_0, which is solved by the standard method.

In accordance with the linearization procedure [4], at the second stage, we find the next approximation $f_1(y) = f(y) - f_0(y)$ using the Newton–Kantorovich scheme [8]

$$A(f) = A(f_0) + A'(f_0)f_1,$$

where the Gâteaux derivative of the operator is found from the definition

$$A'(f_0)f_1 = \lim_{t\to 0}\frac{A(f_0 + tf_1) - A(f_0)}{t}$$

and has the form

$$A'(f_0)f_1 = \Omega\Big[\int_0^1 f_0(y)\Big(\int_0^1 f_1(\eta)\sin(\Omega(\eta-1))\,d\eta\sin(\Omega y) - \int_0^y \sin(\Omega(\eta-y))f_1(\eta)\,d\eta\sin\Omega\Big)\,dy$$

$$+\int_0^1 f_1(y)\Big(\int_0^1 f_0(\eta)\sin(\Omega(\eta-1))\,d\eta\sin(\Omega y) - \int_0^y \sin(\Omega(\eta-y))f_0(\eta)d\eta\sin\Omega\Big)\,dy\Big]. \qquad (3.2)$$

We obtain the Fredholm linear integral equation of the first kind

$$Kf_1 = \int_0^1 K(y,\Omega)f_1(y)\,dy = g_1(\Omega), \qquad \Omega\in[\Omega_1,\Omega_2], \qquad (3.3)$$

in which $g_1(\Omega) = F_*(\Omega) - A(f_0)$, and the kernel $K(y,\Omega)$ is obtained from (3.2):

$$K(y,\Omega) = \Omega\Big[\sin(\Omega(y-1))\int_0^1 f_0(\eta)\sin(\Omega\eta)\,d\eta - \sin\Omega\int_y^1 \sin(\Omega(y-\eta))f_0(\eta)\,d\eta$$

$$+\int_0^1 f_0(\eta)\sin(\Omega(\eta-1))\,d\eta\sin(\Omega y) - \int_0^y \sin(\Omega(\eta-y))f_0(\eta)\,d\eta\sin\Omega\Big]. \qquad (3.4)$$

Calculating the integrals in (3.4) for the linear functions $f_0(y)$ (3.1), after simplifications we obtain the expression for the kernel

$$K(y,\Omega) = a_0[(1-\cos\Omega)(\sin(\Omega(y-1)) - \sin(\Omega y)) + \sin\Omega(2-\cos(\Omega(y-1)) - \cos(\Omega y))]$$
$$-a_1[\cos\Omega\sin(\Omega(y-1)) + \sin(\Omega y) - \sin\Omega(2y - \cos(\Omega(y-1)))], \qquad (3.5)$$

which is a smooth function on $[0,1]\times[\Omega_1,\Omega_2]$. We note that $K(0,\Omega) = K(1,\Omega) = 0$.

Thus, we reduced the problem of determining the function $f_1(y)$ to a Fredholm linear integral equation of the first kind with the smooth kernel (3.4). It is well known that the inversion of this operator is an incorrect problem [7] and requires regularization.

4. To obtain a numerical solution, we regularize the integral equation (3.3) by Tikhonov's method by discretizing the boundary-value problem for the Euler equation and subsequently solving the resulting system of linear algebraic equations [7]. A series of numerical experiments for various functions $f(y)$ was performed. We consider three cases:

Case A. $f(y) = \exp(-\lambda y)$. (4.1)

Case B. $f(y) = b_0y^3 + b_1y^2 + b_2y + b_3$.

Case C. $$f(y) = \begin{cases} \delta_1, & 0\leqslant y\leqslant x_1, \\ \dfrac{\delta_2-\delta_2}{x_3-x_1}(y-x_1)+\delta_1, & x_1\leqslant y\leqslant x_3, \\ \delta_2, & x_3\leqslant y\leqslant 1. \end{cases}$$

For case A for $\lambda = 5$, Fig. 1a shows the graphs of the input characteristic $F_*(\Omega)$ (curve 1), which was calculated using the function (4.1), and characteristic $F_0(\Omega)$ (curve 2), which was calculated for the linear function $f_0(y)$ (3.1) with $a_0 = 0.4924$ and $a_1 = -a_0$, which ensure a minimum of the functional Φ in the domain U_0. Figure 1b shows the graphs of the functions $f(y)$ and $f_0(y)$ (curves 1 and 2) and $f_N(y) = f_0(y) + f_1(y)$ depicted by points. In solving the integral equation (3.3), ten collocation points $N = 10$ were taken to determine $f_1(y)$. It should be noted that the corrections do not exceed 5% with increasing N from 5 to 10; the solution is stable relative to the regularization parameter α varied within the range $[10^{-5}, 0.01]$. In connection with the Remark on the behavior of the kernel (3.5) at the boundary of its domain, it is expedient to choose collocation points so that they do not belong to the boundary $y = 0$ and 1, as done in the numerical calculations.

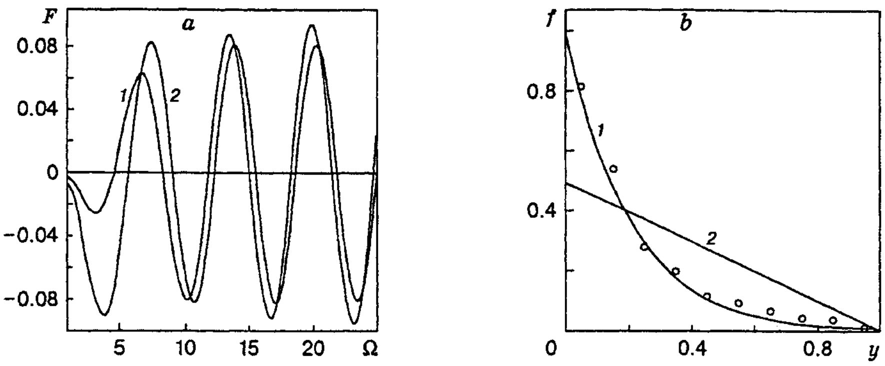

Fig. 1

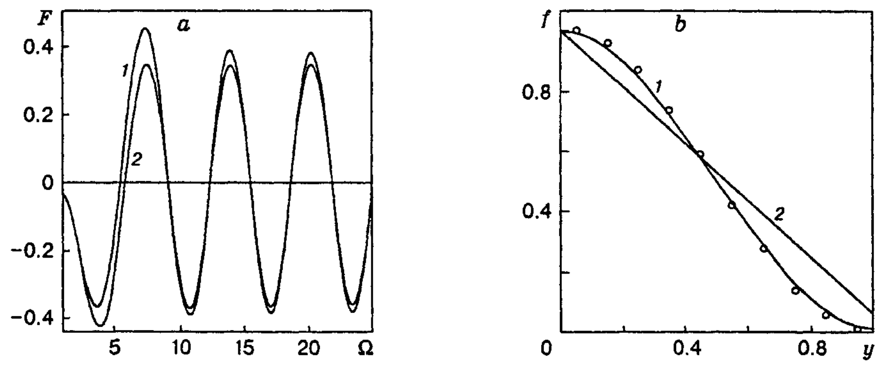

Fig. 2

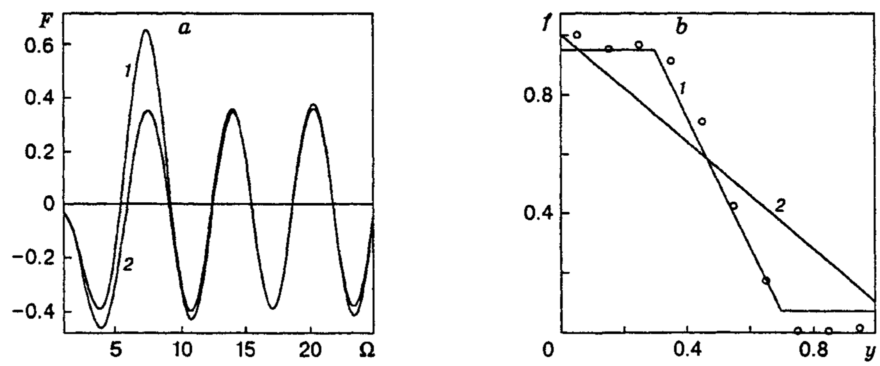

Fig. 3

In case B, the restored curve has an inflection point. The function $f(y)$ has a maximum for $y = 0$ and a minimum for $y = 1$. Moreover, $b_0 = 1.98$, $b_1 = -2.97$, $b_2 = 0$, and $b_3 = 1$. Graphs similar to those described above are represented in Fig. 2. It was found that $a_0 = 1$ and $a_1 = -0.9379$. The relative error in determining the form of $f(y)$ does not exceed 3%, beginning from $N = 5$ for $\alpha \in [10^{-5}, 0.01]$.

In case C, the restored function is piecewise-constant. Numerical calculations show that the error increases when the slope of the linear part of $f(y)$ for $y \in [x_1, x_3]$ approaches $\pi/2$. Dependences similar to those obtained for case A are shown in Fig. 3 for $x_1 = 0.3$, $x_3 = 0.7$, $\delta_1 = 0.95$ and $\delta_2 = 0.07$. The maximum error is 10%. It was found that $a_0 = 1$ and $a_1 = 0.8966$. The calculations were carried out for $N = 10$ and the regularization parameter $\alpha = 0.01$.

It is noteworthy that the above problem of restoration of the function $f(y)$ on the basis of *a priori* physical data concerning its behavior can be considered as an incorrectly formulated problem on a set of special structure and can be solved by the algorithms proposed by Tikhonov et al. [9]. We also used another approach in which the determination of the functions $f_0(y)$ and $f_0(y) + f_1(y)$ is considered as the zero and first steps of an iterative process. As the calculations show, the error at the second step for case B does not exceed 5% compared with 10% at the first step.

As we pointed out above, the initial information for restoration of the piezoelectric modulus $d_{31}(x_1)$ is the function $F(\Omega)$ given by relations (2.1) and (2.2), which can be found experimentally (the conductance of a specimen is measured depending on the vibration frequency). In this connection, it is natural to analyze the effect of the measurement error on the stability of the developed approach.

To this end, for case A, the function $F(\Omega)$ was perturbed and the form was restored by the function $F_2(\Omega)$

$$F_2(\Omega) = F(\Omega) + \varepsilon BH(\Omega) \quad \text{for} \quad \Omega \in [\Omega_1, \Omega_2],$$

where ε is a certain parameter, B is the amplitude of the function $F(\Omega)$, and $H(\Omega)$ is a certain random function such that $|H(\Omega)| \leqslant 1$. The calculations performed for the case where $\varepsilon = 0.1$, $H(\Omega) = \sin(10\Omega)$ for number of collocation points $N = 10$ and $\alpha = 0.5 \cdot 10^{-5}$ show that the approximation error is not greater than 5%, i.e., it does not exceed the error introduced into the function $F_2(\Omega)$.

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